Inapproximability of some art gallery problems

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Abstract

We prove that the three art gallery problems Vertex Guard, Edge Guard and Point Guard for simple polygons with holes cannot be approximated by any polynomial time algorithm with a ratio of $\frac{1-\epsilon}{28}\ln n$, for any $\epsilon > 0$, unless $NP \subseteq TIME(n^{O(\log\log n)})$. We obtain our results by extending and modifying the concepts of a construction introduced in [Eide98].

1 Introduction and problem definition

The art gallery problem of determining how many guards are sufficient to see every point in the interior of an n-wall art gallery room is a classical problem that was originally posed by Klee (see [Hons76]).

The input is a simple polygon P with holes, given as a linked list of n points in the $x-y$-plane. A polygon is called *simple*, if no two nonconsecutive edges of the polygon intersect. We only deal with simple polygons with holes in this paper. Two points *see* each other (in the polygon P) if the line segment connecting the two points does not intersect the exterior of the polygon P.

Many upper and lower bounds on the number of guards needed are known, but far fewer papers deal with the computational complexity of placing guards; the latter is also the topic of this paper. Surveys on the general topic of art galleries include [ORou87], [Sher92], and [Urru98]. [Nils94] contains an overview of what is known about the computational complexity of several art gallery problems.

More specifically, the problem of covering the whole area of a polygon by a minimum number of (possibly overlapping) convex polygons is NP-hard for polygons with holes [ORou83] and for polygons without holes [Culb88]. The problem of covering the whole area of a polygon with a minimum number of (possibly overlapping) star-shaped polygons is equivalent to the Point Guard problem to be defined later. It is NP-hard for polygons with holes [ORou83] and for polygons without holes [Lee86]. The two problems Vertex Guard and Edge Guard (to be defined later) are also NP-hard, even for polygons without holes [Lee86].

We study the following problems in this paper.

- Vertex Guard (VG):
  Given a polygon P (with holes allowed) with n vertices, find a smallest subset S of the set of the vertices of P such that every point on the boundary of the polygon P can be seen from at least one vertex in S. The vertices in S are called vertex-guards.

- Edge Guard (EG):
  Given a polygon P (with holes allowed) with n vertices, find a smallest subset S of edges of P such that every point on the boundary of the polygon P can be seen from at least one point on an edge in S. The edges in S are called edge-guards.

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- Point Guard (PG):
  Given a polygon P (with holes allowed) with n vertices, find a smallest set S of points in the interior of the polygon such that every point on the boundary of the polygon P can be seen from at least one point in S. The points in S are called point-guards.

Note that our definitions differ from the corresponding definitions in [Lee86]: There, the guards must see all of the interior of the polygon, whereas here, they only need to see the boundary. This is done for ease of presentation; nevertheless, it is easy to see that our results carry over to the problems as defined in [Lee86].

We prove our inapproximability results for these problems by constructing a reduction from Set Cover (SC), which is defined as follows:

Set Cover (SC)

Instance:	A finite universe $E = \{e_1, \ldots, e_n\}$ of elements e_i and a collection $S = \{s_1, \ldots, s_m\}$ of subsets s_i with $s_i \subseteq E$.
Problem:	Find $S' \subseteq S$ of minimum cardinality such that every element e_i, $1 \leq i \leq n$, belongs to at least one subset in S'.
Note:	For ease of discussion, let the elements in E and the subsets in S have an arbitrary, but fixed order.

In this paper we extend and modify the ideas and concepts that we used to prove an inapproximability result for the problem of guarding a given 2.5-dimensional terrain with a minimum number of guards at a fixed height [Eide98]. In Section 2, we propose a transformation of a SC-instance into a PG-instance, which is largely similar to a transformation proposed in [Eide98]. We show that our transformation has all the desired properties in Section 3. The transformation of the result, back to the original problem is presented in Section 4. We prove that our transformation is polynomial in Section 5. We prove our inapproximability result for PG in Section 6 and show that it carries over to VG and to EG in Section 7. We draw conclusions in Section 8.

2 Construction of the reduction

We prove our inapproximability result for the Point Guard (PG) problem with holes by showing how to construct an instance of PG for every instance of Set Cover (SC). In order to make this paper self-contained, we present here in detail the construction, based on a similar construction in [Eide98].

We construct a polygon in the $x-y$-plane; Figure 1 shows this construction. For each set $s_i, i = 1, \ldots, m$, place on the horizontal line $y = y_0$ the point $((i-1)d', y_0)$. This places a sequence of points from left to right, one point per set s_i for $i = 1, \ldots, m$, with d' a constant distance between two adjacent points. For ease of description, call the i-th point s_i. For each element $e_i \in E$, place on the horizontal line $y = 0$ two points $(D_i, 0)$ and $(D'_i, 0)$, with $D'_i = D_i + d$ for a positive constant d and $D_1 \geq 0$. Arrange the points from left to right for $i = 1, \ldots, n$, with distances $d_i = D_{i+1} - D'_i$ to be defined later. Call the points also D_i and D'_i, for $i = 1, \ldots, n$.

For every element e_i, draw a line g through s_j and D_i, where s_j is the first set of which e_i is a member. Also draw a line g' through s_l and D'_i, where s_l is the last set of which e_i is a member [1]. Let the intersection point of g and g' be I_i. Then draw line segments from every s_k that has e_i as a member to D_i and to D'_i.

Two lines connecting an element e_i with a set s_j form a cone-like feature; the area between these two lines will therefore be called a *cone*. Call the triangle $D_i I_i D'_i$ a *spike*. The point I_i

[1] We assume w. l. o. g. that each element is a member of at least two sets.

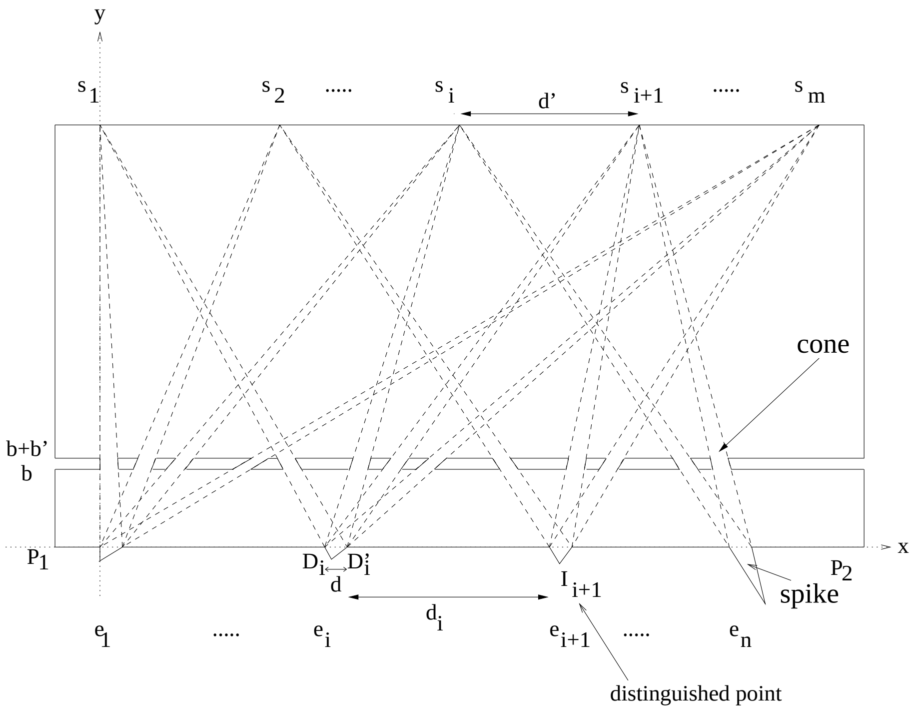

Figure 1: Basic construction

of each spike plays a special role and is therefore called the *distinguished point* of the spike.

We have only constructed one part of the polygon thus far: Among all the lines described, only the spikes and the line segments of the horizontal line $y = 0$ that are between adjacent spikes are part of the polygon boundary, all other lines merely help in the construction.

In our construction the guards of an optimum solution will have to be placed at the points s_j, therefore we need to make sure that a guard at s_j sees the spikes of only those elements e_i that are a member of the set s_j. This is achieved by introducing a "barrier"-line at $y = b$, see Figure 1. Only line segments on the horizontal line $y = b$ that are outside the cones are part of the polygon boundary. We draw another barrier-line with distance b' from the first barrier at $y = b + b'$. Define holes of the polygon by connecting endpoints of line segments of the two barrier lines that belong to the same cone-defining line. We call the area between the two lines at $y = b$ and $y = b + b'$ (including all holes) the *barrier*. Thus, the barrier contains a small part of all cones.

As a next step in the construction of the polygon, draw a vertical line segment at $x = -d''$, where d'' is a positive constant, from $y = 0$ to $y = y_0$. This line segment is part of the polygon boundary except for the segment between the two barrier lines.

Choose the coordinates (to be shown later) such that the rightmost spike is farther right than the rightmost set, i.e. $D'_n > s_m$ (for reasons of space, we violated this condition in Figure 1), and draw another vertical line segment from $y = 0$ to $y = y_0$ at $x = D'_n + d''$, again taking a detour at the barrier. The boundary lines of the polygon defined so far are shown as solid lines in Figure 1. It is important to note that the cones, drawn as dashed lines in the figures, are not part of the polygon boundary.

The thickness b' of the barrier is defined such that all segments of all holes except for those on the line $y = b + b'$ can be seen from two guards at $P_1 = (-d'', 0)$ and $P_2 = (D'_n + d'', 0)$. To achieve this, the thickness b' is determined by intersecting (for each pair of adjacent holes) a line from P_1 through the lower right corner of the left hole (of the pair of adjacent holes) with a line from P_2 through the lower left corner of the right hole as shown in Figure 2. Now, the

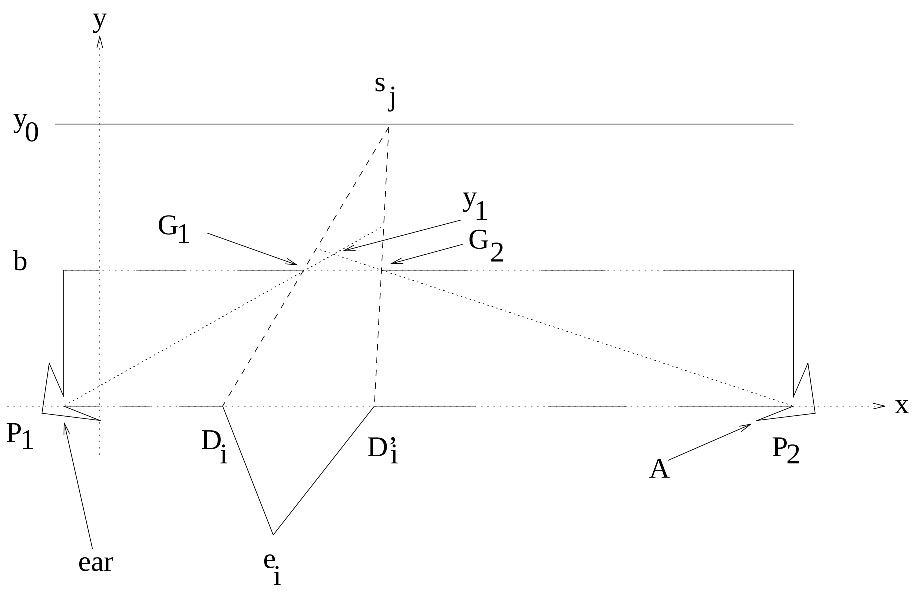

Figure 2: Thickness of the barrier and ears

barrier line $y = b + b'$ is defined to go through the lowest of all these intersection points (they are indeed all at the same height, by arguments with similar triangles).

In order to simplify our proof, we attach another feature, which is called an *ear*, to the corners P_1 and P_2, forcing one guard each to P_1 and P_2. Ears are shown in Figure 2.

We set the parameters of the reduction as follows: Let d' and y_0 be arbitrary positive constants. Let d and b be positive constants as well, where $d = \frac{d'}{4}$ and $b = \frac{5}{12}y_0$. We let $b' = \frac{\frac{35}{12^2}y_0}{-4^{l-1}m^{l-1}+2\sum_{i=0}^{l-1}4^i m^i + 2\frac{d''}{d} - \frac{19}{12}}$ and $D_l = -4^{l-1}m^{l-1}d - d + 2d\sum_{i=0}^{l-1}4^i m^i$ for $l = 1, \ldots, n$.

3 Properties of the reduction

In order for the reduction to work, it is necessary that at no point a guard sees the distinguished points of the spikes of three or more elements e_i unless there is a set s_j that contains all three elements.

A guard that is placed at some point with y-value between 0 and b, i.e., between the barrier and the spikes, sees at most one such distinguished point, provided the barrier is placed such that no cones of two different elements intersect in the area below the barrier.

We must ensure that a guard that is placed at some point with y-value between $b + b'$ and y_0 does not see the distinguished points of three or more elements unless there is a set that contains all three elements. In order to do this, we introduce the notion of *extended cones* as shown in Figure 3. The extended cone is the area in the rectangle $D_i, D'_i, s_j + a, s_j - a$. Point $s_j - a$ is defined as the intersection point of the line $y = y_0$ with the line from D'_i through the lower right corner of the left of the two holes which contain a part of the cone from set s_j and element e_i. Point $s_j + a$ is defined accordingly. It will be easy to see that points $s_j - a$ and $s_j + a$ are both at a constant distance a from point s_j (see proof of Lemma 3.1).

For a guard between the two horizontal lines $y = b + b'$ and $y = y_0$, in order to see the distinguished point of the spike of e_i, it must lie in the area of the triangle defined by the points h_1, h_2 and I_i (or, of course, in the corresponding triangle of any other point $s_{j'}$ with $e_i \in s_{j'}$). In order to keep the analysis simple, we will argue with the extended cones rather than the triangles. If no three extended cones from three different elements and three different

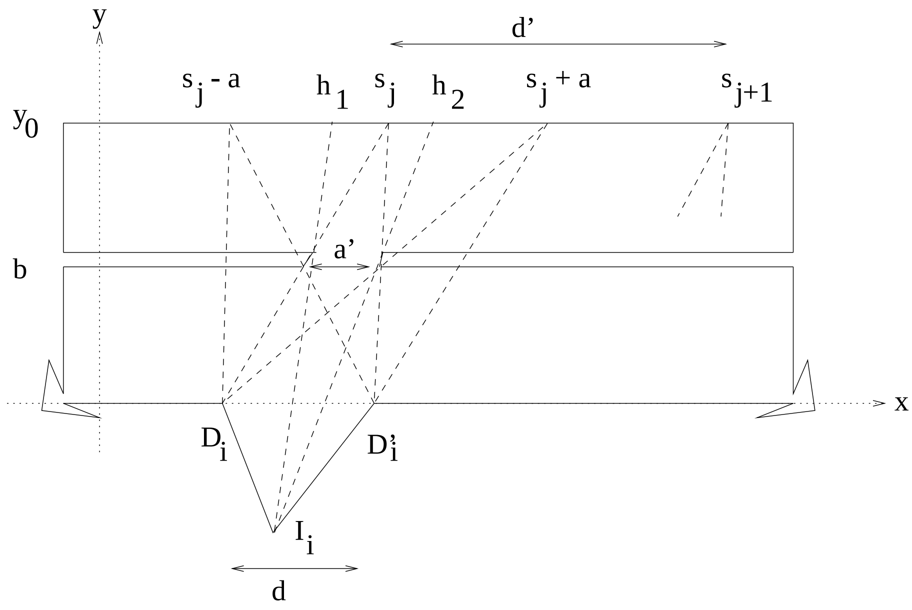

Figure 3: Intersection of three cones

sets intersect in this area, then it is ensured that there exists a pair of setpoints such that all distinguished points that a guard in this area sees can also be seen from at least one of the setpoints of the pair. (It is, of course also possible that a single setpoint sees all the distinguished points that a guard in this area sees).

A guard that is placed at some point with y-value less than 0, sees the distinguished point of at most one spike, if it is ensured that no two spikes intersect.

Thus, we need to prove the following:

- No three extended cones from different elements and sets intersect.
- The barrier is such that all intersections of cones from the same element e_i are below b (to ensure that the view of the points s_j is blocked appropriately) and such that all intersections of cones from different elements are above $b + b'$ and such that all of the barrier except for the line segments at $y = b + b'$ can be seen from at least one of two guards at P_1 and P_2.
- No two spikes intersect.

3.1 No three extended cones from different elements and sets intersect

Lemma 3.1 *For $e_l \in s_{l'}$, let:*

$$D_l \quad \geq \quad \max\left(\frac{s_{i'} - s_{l'}}{s_{i'} - s_{j'} - 2a}(D_j + d - D_i) + D_i + d\right)$$

where the maximum is taken over all $e_i \in s_{i'}$ and $e_j \in s_{j'}$, for which $i < j < l$ and $l' < j' < i'$ holds. Then the three extended cones from e_l to $s_{l'}$, from e_i to $s_{i'}$ and from e_j to $s_{j'}$, with $i < j < l$ do not have a common intersection point.

Proof: Assume that the positions of the elements, i.e., the values D_v, have been set for all $v < l$ such that no three extended cones (connecting three different sets with three different elements) intersect. We show how to set D_l such that no three extended cones intersect; see

Figure 4. Let S be an intersection point with maximum y-value among the two extended cones connecting the elements e_i and e_j with the (different) points $s_{j'}$ and $s_{i'}$.

In order to ensure that our construction is feasible, S must lie in the area between y_0 and the barrier. Let S_x be the x-value of S. Then, $S_x < y_0$. To see this, note that this is equivalent to saying that $s_{j'} + a < s_{i'} - a$ (see Figures 3 and 4), which is a weaker condition than $s_{j'} + a < s_{j'+1} - a$. Now, $s_{j'} + a < s_{j'+1} - a$ is equivalent to $2a < d'$. We express a as a function of y_0, b and d using the similarity of triangles. Note that $\frac{a'}{d} = \frac{y_0 - b}{y_0}$ and $\frac{b}{y_0} = \frac{a'}{a}$. Thus, we get $a = \frac{1-b}{b} d$. Using this result in $2a < d'$, we obtain:

$$b \quad > \quad \frac{2}{\frac{d'}{d} + 2} y_0,$$

which is equivalent to $b > \frac{1}{3} y_0$, since $d = \frac{d'}{4}$. This inequality for b is satisfied, since $b = \frac{5}{12} y_0 > \frac{1}{3} y_0$.

For each set $s_{l'}$ of which e_l is a member, draw a line through S, determine where it intersects the line $y = 0$ and let $D^S_{l,l'}$ be the x-value of this intersection point. Let $D^S_l = \max_{l'} D^S_{l,l'}$ be the maximum x-value of all intersection points defined this way. For any pair of extended cones in "inverse position" to the left of e_l, with which an extended cone at e_l forms a "triple inversion", compute the corresponding D^S_l and let $D^{\max}_l$ be the maximum D^S_l. Finally, we let $D_l = D^{\max}_l + d$ to ensure that no three extended cones have one common intersection point at some point S. Figure 4 shows the situation for an intersection and explains the notation. The point S is the intersection point of the lines from $s_{i'} - a$ to D_i and from $s_{j'} + a$

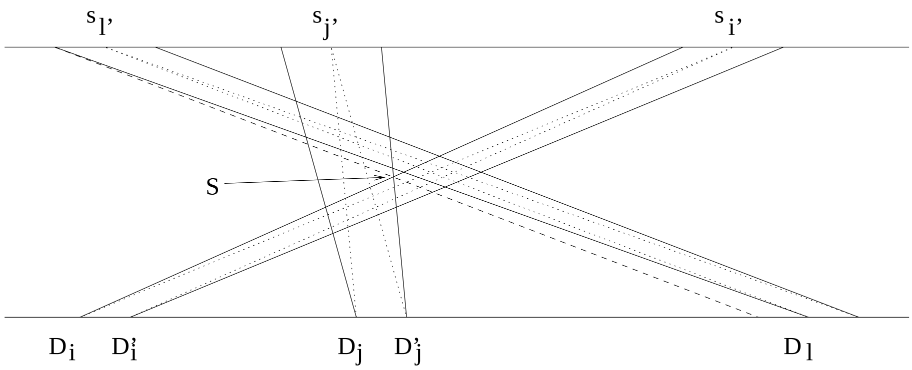

Figure 4: Intersection of three cones

to D'_j. Simple geometric calculations yield: $S = ((1 - t_1)(s_{i'} - a) + t_1 D_i, y_0(1 - t_1))$ with $t_1 = \frac{s_{i'} - s_{j'} - 2a}{D'_j - D_i + s_{i'} - s_{j'} - 2a}$. Draw the line from $s_{l'} - a$ to S, and simple geometric calculations show: $D^S_{l,l'} = \frac{s_{i'} - s_{l'}}{s_{i'} - s_{j'} - 2a}(D_j + d - D_i) + D_i$. The lemma follows. □

Lemma 3.1 implies:

$$\begin{aligned} \max\left(\frac{s_{i'} - s_{l'}}{s_{i'} - s_{j'} - 2a}(D_j + d - D_i) + D_i + d\right) &\leq \max\left(\frac{md'}{d' - 2a}(D_j + d) + d\right), \forall j < l \\ &\leq 4m(D_{l-1} + d) + d, \end{aligned}$$

where we have used $a = \frac{1-b}{b} d = \frac{7}{5} d$ and $d' = 4d$ in the last step. Now, let $D_l = 4m(D_{l-1} + d) + d$. It is easy to see that this is consistent with our definition of D_l, since:

$$-4^{l-1} m^{l-1} d - d + 2d \sum_{i=0}^{l-1} 4^i m^i = 4m((-4^{l-2} m^{l-2} d - d + 2d \sum_{i=0}^{l-2} 4^i m^i) + d) + d$$

3.2 The barrier is in good position

Lemma 3.2 *Any two cones that belong to the same element e_i intersect only at points with y-values at most $y_0 \frac{d}{d+d'}$.*

Proof: Let e_i be a member of s_j and s_l and $s_j < s_l$. The intersection point of the lines and from s_l to D_i is the point in the intersection area of the two cones that has the largest y-value. The lemma follows by geometric calculations. $\square$

Lemma 3.3 *Any two cones that belong to elements e_i, e_j, respectively, with $i < j$, intersect only at points with y-values at least $y_0 \frac{d_i}{d_i + md'}$.*

Proof:

Let e_i be a member of $s_{i'}$ and let e_j be a member of $s_{j'}$, also let $D_i < D_j$ and $s_{j'} < s_{i'}$. Exactly then, the corresponding two cones intersect. The intersection point of the lines from $s_{j'}$ to D_j and from $s_{i'}$ to D_i' is the point in the intersection area of the two cones with minimum y-value. The lemma follows by some geometric calculations. $\square$

Lemma 3.4 *Let $b' = \frac{bd(y_0 - b)}{y_0(p_2 - p_1) - d(y_0 - b)}$, where p_1 and p_2 are the x-values of the points P_1 and P_2. Then all of the barrier including the segments of the cones except for the segments at $y = b + b'$ can be seen from the two guards at P_1 and P_2.*

Proof: Let $e_i \in s_j$ and let G_1 and G_2 be the two points where this cone intersects with the barrier line $y = b$ (see Figure 2). We find an expression for y_1, which is the y-value of the intersection point of the two lines from P_1 to G_1 and from P_2 to G_2, and the lemma follows by simple geometric calculations. $\square$

If we substitute $b = \frac{5}{12} y_0$ and $p_2 - p_1 = -4^{n-1} m^{n-1} d - d + 2d \sum_{i=0}^{n-1} 4^i m^i + d'' - (-d'') = -4^{n-1} m^{n-1} d - d + 2d \sum_{i=0}^{n-1} 4^i m^i + 2d''$ in the equation for b', we obtain:

$$b' \quad = \quad \frac{\frac{35}{12^2} y_0}{-4^{n-1} m^{n-1} 2 \sum_{i=0}^{n-1} 4^i m^i + 2\frac{d''}{d} - \frac{19}{12}}$$

A simple calculation shows that $b' < \frac{y_0}{12}$, if $m \geq 2$ and $n \geq 2$, which must be the case since there were no intersections otherwise.

Because of $d = \frac{d'}{4}$ and because of Lemma 3.2, any two cones from the same element intersect only at points with y-value at most $\frac{1}{5} y_0$ which is less than b. Because of $d_i \geq md'$ for all d_i and because of Lemma 3.3, any two cones from different elements intersect only at points with y-value at least $\frac{1}{2} y_0$, which is at most $b + b'$.

3.3 Spikes of two elements do not intersect

Lemma 3.5 *The spikes of any two elements do not intersect.*

Proof:

We determine the x-value x_l of the point I_l in the spike of e_l. Note that $x_l > D_l$. Simple calculations show that $x_l \leq 2D_l$. Since $D_{l+1} = 4m(D_l + d) + d$ and since we can assume that $m \geq 1$, the lemma follows. $\square$

4 Transformation of the solution

Given a solution of the PG-instance, i.e. the coordinates of r guards $g_1, \ldots, g_r$, proceed as follows to obtain a solution for the SC-instance:

For each guard g_i, determine the set h_i of elements e_j of which the guard g_i sees the distinguished point of the spike.

Since no three extended cones from three different elements and three different sets intersect in the area above $y = b + b'$ by our construction, there exists a pair of sets (s_k, s_l) for each guard g_i such that $h_i \subseteq s_k \cup s_l$. Determine such a pair of sets for each guard g_i and add the sets to the solution of the SC-instance.

5 The reduction is polynomial

Note that d, d', y_0, h, b are all constants in our reduction. The values for b' and for all D_i are computable in polynomial time and can be expressed with $O(n \log m)$ bits.

Therefore, the construction of the polygon can be done in time polynomial in the size of the input SC-instance, since it only produces a polynomial number of points that each can be computed in polynomial time and each take at most $O(n \log m)$ bits to be expressed.

It is obvious that the backtransformation runs in polynomial time, since it only involves determining whether two points see each other and finding pairs of sets.

6 An inapproximability result for PG

In order to prove a strong inapproximability result, we need some properties of the reductions that were used to prove inapproximability results for Set Cover. There are two inapproximability results for SC which prove that for some constant factor $c > 0$ an approximation ratio of $c \ln n$ cannot be achieved by any polynomial algorithm for SC, unless $NP = P$ [RaSa97] [ArSu97]. On the other hand, a result by Feige [Feig96] shows that SC cannot be approximated by a polynomial algorithm with a ratio of $(1 - \epsilon) \ln n$ for any $\epsilon > 0$, unless $NP \subseteq TIME(n^{O(\log\log n)})$. In this paper, we take advantage of a property of the SC-instances produced in the reduction by Feige in [Feig96]. We proved this property in [Eide98].

Lemma 6.1 *[Eide98] Let N be the number of elements and let M be the number of sets in any SC-instance produced by the reduction in [Feig96]. Then $M \leq N^5$ holds.*

Now consider only those SC-instances that are produced in the reduction in [Feig96] and their corresponding PG-instances. Then, an approximation ratio of $(1 - \epsilon) \ln n$ for any $\epsilon > 0$ cannot be guaranteed by a polynomial algorithm for those SC-instances unless $NP \subseteq TIME(n^{O(\log\log n)})$, since this would imply that the NP-hard problem 5-Occurrence-3-Sat, which is the problem reduced to Set Cover in [Feig96], could be solved efficiently.

Lemma 6.2 *Consider the promise problem of SC (for any $\epsilon > 0$), where it is promised that the optimum solution OPT is either less or equal to c or greater than $c(1-\epsilon)\ln n$ with c, n and OPT depending on the instance I. This problem is NP-hard under slightly superpolynomial reductions [2]. Then, we have for the optimum value OPT' of the corresponding PG-instance I', that OPT' is either less or equal to $c + 2$ or greater than $\frac{c+2}{28}(1 - \epsilon) \ln |I'|$. More formally:*

$$OPT \leq c \implies OPT' \leq c + 2 \tag{1}$$

$$OPT > c(1 - \epsilon) \ln n \implies OPT' > \frac{c+2}{28}(1 - \epsilon) \ln |I'| \tag{2}$$

[2] i.e. reductions that take $n^{O(\log \log n)}$ time; see also the notion of quasi-NP-hardness in [ArLu96]

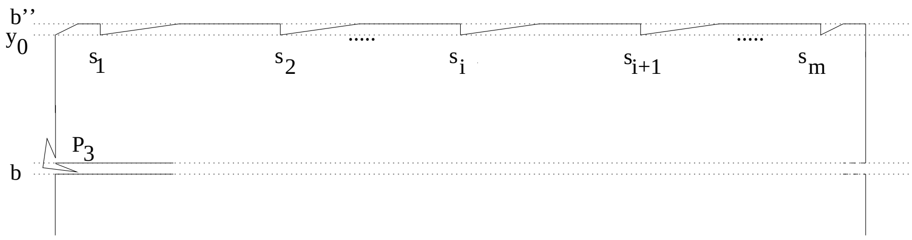

Figure 5: Polygon for VG and EG

Proof: The implication in (1) is trivial, since, given a solution of the SC-instance I of size c, we position a guard at each point s_j in the corresponding PG-instance I', if the set s_j is in the solution of I, and we position two additional guards at points P_1 and P_2 in I', which see the ears and the barrier from below.

We prove the contraposition of (2), i.e.:

$$OPT' \leq \frac{c+2}{28}(1-\epsilon)\ln|I'| \implies OPT \leq c(1-\epsilon)\ln n$$

Observe that, if we are given a solution of I' with k guards, we can obtain a solution of I with at most $2k$ guards by performing the procedure described in Section 4. Therefore:

$$\begin{aligned} OPT &\leq 2\frac{c+2}{28}(1-\epsilon)\ln|I'| && (3) \\ &\leq 2\frac{c+2}{28}(1-\epsilon)\ln n^7 && (4) \\ &\leq 2\cdot 7\frac{2c}{28}(1-\epsilon)\ln n && (5) \\ &\leq c(1-\epsilon)\ln n && (6) \end{aligned}$$

where we used $|I'| \leq n^7$ to get (4), which is true because the polygon of I' consists of n spikes, less than nm holes and two ears. Therefore, the polygon consists of less than $k(nm+n+2)$ points, where k is a small constant. Since $k < n$ and $m < n^5$ (see Lemma 6.1), $|I'| \leq n^7$. We used $2c \geq c+2$ to get to (5).

□

Lemma 6.2 establishes our first main result (see [ArLu96] for details on this proof technique).

Theorem 6.3 *The* POINT GUARD *problem for polygons with holes cannot be approximated by a polynomial time algorithm with an approximation ratio of* $\frac{1-\epsilon}{28}\ln n$ *for any* $\epsilon > 0$*, where* n *is the number of the polygon vertices, unless* $NP \subseteq TIME(n^{O(\log\log n)})$.

7 Inapproximability results for VG and EG

A slight modification of the polygon as indicated in Figure 5, where $b'' = y_0 + b'$, allows us to prove the corresponding theorems for VG and EG.

Theorem 7.1 *The* VERTEX GUARD *problem for polygons with holes cannot be approximated by a polynomial time algorithm with an approximation ratio of* $\frac{1-\epsilon}{28}\ln n$ *for any* $\epsilon > 0$*, where* n *is the number of polygon vertices, unless* $NP \subseteq TIME(n^{O(\log\log n)})$.

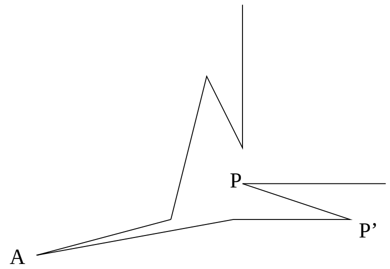

Figure 6: Ear for EG

Proof: The proof is almost identical to the proof for PG, except that instead of two additional guards at P_1 and P_2 we have a third additional guard at P_3 (see Figure 5). This additional guard means that we need to replace $c+2$ by $c+3$ in the proof of Lemma 6.2. In addition, we get a slightly stronger condition, namely $2c \geq c+3$, to obtain the inequality at (5). □

Theorem 7.2 *The* EDGE GUARD *problem for polygons with holes cannot be approximated by a polynomial time algorithm with an approximation ratio of* $\frac{1-\epsilon}{28}\ln n$ *for any* $\epsilon > 0$*, where* n *is the number of polygon vertices, unless* $NP \subseteq TIME(n^{O(\log\log n)})$.

Proof: The proof is almost identical to the proof for PG with the additional information from the proof of Theorem 7.1. Note that in the case of EG all guards are edges. The proofs carry over effortlessly, except for the shape of the ears. We use the ears as shown in Figure 6. Point A of the ear can be seen from the edge-guard PP'. □

8 Conclusion and remarks

We have shown that the art gallery problems POINT GUARD, VERTEX GUARD and EDGE GUARD for simple polygons with holes cannot be approximated by any polynomial time algorithm with a ratio of $\frac{1-\epsilon}{28}\ln n$ for any $\epsilon > 0$ unless $NP \subseteq TIME(n^{O(\log\log n)})$.

It is obvious from the construction of the polygon and the distinguished points of the spikes that our results carry over to the variants of the problem where all of the interior of the polygon needs to be seen from at least one guard rather than only the boundary.

Our result characterizes the approximability of VG and EG for polygons with holes exactly up to a constant factor, since polynomial time approximation algorithms for EG and VG with a logarithmic ratio exist [Ghos87]. No approximation algorithms are known for PG.

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