Positioning Guards at Fixed Height above a Terrain – an Optimum Inapproximability Result*

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Abstract. We study the problem of minimizing the number of guards positioned at a fixed height h such that each triangle on a given 2.5-dimensional triangulated terrain T is completely visible from at least one guard. We prove this problem to be NP-hard, and we show that it cannot be approximated by a polynomial time algorithm within a ratio of $(1-\epsilon)\frac{1}{35}\ln n$ for any $\epsilon > 0$, unless $NP \subseteq TIME(n^{O(\log\log n)})$, where n is the number of triangles in the terrain. Since there exists an approximation algorithm that achieves an approximation ratio of $\ln n+1$, our result is close to the optimum hardness result achievable for this problem.

1 Introduction and Problem Definition

We study the problem of positioning a minimum number of guards at a fixed height above a terrain. The *terrain* is given as a finite set of points in the plane, together with a triangulation (of its convex hull), and a height value is associated with each point (a triangulated irregular network (TIN), see e.g. [5]). The TIN defines a bivariate, continuous function; this surface in space is also called a 2.5-dimensional terrain. A *guard* is a point in space above the terrain. A guard can see a point of the terrain if the straight line segment between the guard and the point does not intersect the terrain. That is, a particular guard point can see some parts of the terrain, while others might be hidden. We ask for a smallest set of guards at a fixed height that together see the whole terrain. More precisely, we study a problem we call TERRAIN COVER (TC), where the input is a 2.5-dimensional terrain, given as a TIN, and a height h, and where the goal is to find a smallest set of guard points at height h such that every triangle can be seen from at least one guard. We assume h to be such that all points in the terrain are below h. Note that our requirement that each triangle be completely covered by one guard is a particular version of the problem, different from the version in which a triangle may also be covered by several guards together, with each guard covering only a part of the triangle. This problem models a question that arises after the liberalization of the telecommunications market in Switzerland. Companies are planning to place communication stations above the Swiss mountains in extremely low position - balloons at a height of 20 km

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above sea level - and hold them in geo-stationary position. If we simply model electromagnetic wave propagation at high frequencies (GHz) by straight lines of sight, ignoring reflection and refraction, the TERRAIN COVER problem asks for a cover with the smallest number of balloons.

Related problems have been considered previously. Guarding a polygon has been studied in depth; for an overview, see the surveys [6], [7], [8] and [9], or any textbook on computational geometry. More specifically, [2] deals with optimum guarding of polygons and monotone chains. When guards can only be positioned directly on a given 1.5-dimensional terrain that must be completely covered, the problem of finding the minimum number of guards is NP-hard. A shortest watchtower for a given 2.5 dimensional terrain, i.e., a single guard position closest to the terrain (in vertical direction) that sees all of the terrain, can be found in $O(n \log n)$ time [10]. The related problem of finding the lowest watchtower, i.e., a single guard position with smallest height value that sees all of the terrain, can be solved in linear time using linear programming. Some upper and lower bounds on the number of guards needed to guard a terrain, when guards can only be positioned at the vertices of a 2.5-dimensional terrain, have been derived [1]. Our problem of guarding (covering) a terrain at a fixed height has not been studied in the literature so far. However, a previous result [6] implies that the TERRAIN COVER problem for a 1.5-dimensional terrain can be solved in linear time.

We proceed as follows in this paper. We first propose an approximation algorithm for the TERRAIN COVER problem for a 2.5-dimensional terrain that guarantees an approximation ratio of $\ln n + 1$, where n is the number of triangles in the TIN, and ln is the natural logarithm. We do so by showing (in Sect. 2) how a solution of the SET COVER problem can be used to solve TERRAIN COVER approximately. In SET COVER (SC), we are given a finite universe $E = \{e_1, \cdots, e_n\}$ of elements e_i and a collection of subsets $S = \{s_1, \cdots, s_m\}$ with $s_i \subseteq E$, and we need to find a subset $S' \subseteq S$ of minimum cardinality such that every element e_i belongs to at least one member in S'. For ease of discussion, let E and S have an arbitrary, but fixed order.

Our proposed approximate solution brings up the question of whether this approximation is the best possible. It is the main contribution of this paper to show that indeed a better approximation is impossible, up to a constant factor, unless NP has $n^{O(\log \log n)}$-time deterministic algorithms. To this end, we propose a reduction from SC to TC (Sect. 3). Our reduction constructs a (planar) polygon with holes from a given instance of SC; in a second step a terrain is built by turning the inside of the polygon into a canyon, with steep walls on the polygon boundary and columns for the holes. Recall that a reduction from the problem SC to the problem TC is a pair (f, g) of two functions such that for any instance I of SC, $f(I)$ is an instance of TC and such that for every feasible solution z of $f(I)$, $g(z)$ is a feasible solution of I. Furthermore, if z' is an optimum solution of $f(I)$, then $g(z')$ is an optimum solution of I. In addition, both functions must be computable in time polynomial in the size of the SC-instance, i.e. polynomial in $|I|$. We show that the reduction has all the desired properties and can be computed efficiently (Sect. 4). In Sect. 5, we show how an inapproximability result for Set Cover by Feige [4] carries over with the proposed construction to Terrain Cover. Precisely, we prove that Terrain Cover cannot be approximated with ratio $((1-\epsilon)/35)\ln n$, for any $\epsilon > 0$, by a polynomial time algorithm, unless $NP \subseteq TIME(n^{O(\log\log n)})$. Section 6 contains some concluding remarks and discusses implications for other problems.

2 An Approximation Algorithm

TC can be approximated with a ratio $\ln n + 1$, where n is the number of triangles, by constructing an SC-instance for a given TC-instance as follows: Each triangle is an element of the SC-instance. For each triangle determine the area on the plane $z = h$ from where the triangle is fully visible. This area is a polygon of descriptional complexity $O(n^2)$, that can be computed in time $O(n^4)$ by interpreting the points of the polygon as special points of an arrangement. At each point, where two of these polygons intersect, determine which triangles are visible from this point and define the set of visible triangles as one set for SC. There are $O(n^6)$ such intersections. Now solve the SC-instance approximately, by applying the well-known greedy algorithm for SC: add to the solution the set that covers a maximum number of elements not yet covered. This solution is not more than $\ln n + 1$ times larger than the optimum solution for SC. To see that this reduction is approximation-ratio-preserving, consider that the n polygons partition the plane $z = h$ into cells. Observe that the set of visible triangles is the same throughout the area of a cell. On the boundary of the cell, however, a few more triangles might be visible since the boundary may be part of the visibility area of another triangle. Therefore, any solution of the TC-instance can be transformed to a solution of the SC-instance by moving guards that are in the interior of a cell to an appropriate intersection point on the boundary of the cell.

3 Construction of the Reduction

In order to prove our inapproximability result for Terrain Cover (TC), we show how to construct an instance of TC for every instance of Set Cover (SC), i.e., we describe the function f of the reduction. The construction first leads to a polygon (with holes); we then construct a terrain by simply letting the area inside the polygon have height 0 and letting the area outside the polygon (including the holes) have height h', where h' is slightly less than h.

We construct the polygon in the $x-y$-plane; Figure 1 shows this construction. For the sequence of sets $s_1, \cdots, s_m$, place on the horizontal line $y = y_0$ the sequence of points $((i-1)d', y_0)$ from left to right for $i = 1, \cdots, m$, with d' a constant distance between two adjacent points. For ease of description, call the i-th point s_i. For each element $e_i \in E$, place on the horizontal line $y = 0$ two points $(D_i, 0)$ and $(D'_i, 0)$, with $D'_i = D_i + d$ for a positive constant d. Arrange the points from left to right for $i = 1, \cdots, n$, with distances $d_i = D_{i+1} - D'_i$ to be defined later. Call the points also D_i and D'_i, for $i = 1, \cdots, n$.

For every element e_i, draw a line g through s_j and D_i, where s_j is the first set of which e_i is a member. Also draw a line g' through s_l and D'_i, where s_l is the last set of which e_i is a member [1]. Let the intersection point of g and g' be I_i. Then draw line segments from every s_k that has e_i as a member to D_i and to D'_i. Two lines connecting an element e_i with a set s_j form a cone-like feature; the area between these two lines will therefore be called a *cone*. Call the triangle $D_i I_i D'_i$ a *spike*. We have only constructed one part of the polygon thus far: Among all the lines described, only the spikes and the line segments of the horizontal line $y = 0$ that are between two spikes are part of the polygon boundary, all other lines merely help in the construction. In our construction the guards of an optimum solution will have to be placed at the points s_j, therefore we need to make sure that a guard at s_j only sees the spikes of those elements e_i that are a member of the set s_j. This is achieved by introducing a barrier-line at $y = b$, see Fig. 1. Only line segments on the horizontal line $y = b$ that are outside the cones are part of the polygon. We draw another barrier-line with distance b' from the first barrier at $y = b + b'$. Define holes of the polygon by connecting endpoints of line segments of the two barrier lines that belong to the same cone-defining line. We call the area between the two lines at $y = b$ and $y = b + b'$ (including all holes) the *barrier*. Thus, the barrier contains a small part of all cones.

As a next step in the construction of the polygon, draw a vertical line segment at $x = -d''$, where d'' is a positive constant, from $y = 0$ to $y = y_0$. This line segment is part of the polygon boundary except for the segment between the two barrier lines. Assume that the rightmost spike is farther right than the rightmost set, i.e. $D'_n > s_m$, and draw another vertical line segment from $y = 0$ to $y = y_0$ at $x = D'_n + d''$, again taking a detour at the barrier. The boundary lines of the polygon defined so far are shown as solid lines in Fig. 1. The thickness b' of the barrier is defined such that all segments of all holes except for those on the line $y = b + b'$ are visible from two guards at $P_1 = (-d'', 0)$ and $P_2 = (D'_n + d'', 0)$. To achieve this, the thickness b' is determined by intersecting (for each pair of adjacent holes) a line from P_1 through the lower right corner of the left hole (of the pair of adjacent holes) with a line from P_2 through the lower left corner of the right hole as shown in Fig. 2. Now, the barrier line $y = b + b'$ is defined to go through the lowest of all these intersection points. (We will show in Sect. 4 that all intersection points actually lie on this line.)

In order to simplify our proof, we attach another feature, which is called an ear, to the corners P_1 and P_2, forcing one guard each to P_1 and P_2. Ears are shown in Fig. 2. Our construction aims at forcing guards for element spikes at points for sets, but there is a potential problem if a guard is placed in an area where two cones intersect: Such a guard may see the spikes of two elements that are not both a member of the same set. Therefore, we duplicate the whole construction by flipping it over at the horizontal line $y = y_0$. The result is shown

[1] We assume w. l. o. g. that each element is a member of at least two sets.

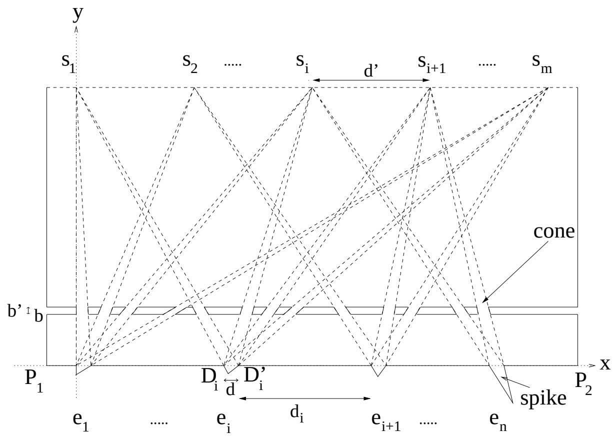

Fig. 1. Basic construction

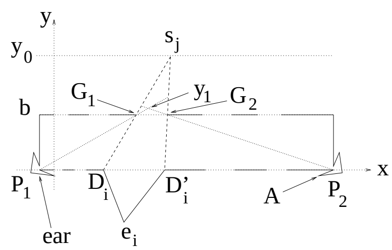

Fig. 2. Thickness of the barrier and ears

in Fig. 3. We denote the mirror image spike of e_i by e_i' and the mirror image points of P_1, P_2 by P_1', P_2'. It is important to note that the cones, drawn as dashed lines in the figures, are not part of the polygon.

Given the polygon, the terrain is defined by placing the interior of the polygon at height $z = 0$ and the exterior at height $z = h'$, with $h = h' + \delta$ and δ a small positive constant, with vertical walls along the polygon boundary. The latter is for simplicity of the presentation only; the terrain can easily be modified to have steep, but not vertical, walls such that the terrain actually is a continuous function in two variables and such that our proofs still work. The resulting terrain is triangulated in such a way that the total number of triangles is polynomial in the input size, i.e., the size of the SC-instance, and such that each spike is triangulated as one triangle only. We set the parameters of the reduction as follows: d' and y_0 are arbitrary positive constants; d and b are positive constants as well, where $d = \frac{d'}{2}$ and $b = \frac{5}{12}y_0$. We let $b' = \frac{\frac{35}{12^2}y_0}{2\sum_{i=1}^{n} m^i + 2\frac{d''}{d} - \frac{7}{12}}$ and $D_l = d + 2d\sum_{i=1}^{l} m^i$ for $l = 1, \cdots, n$. We will prove in Sect. 4 that the reduction is feasible and runs in polynomial time with these parameter values.

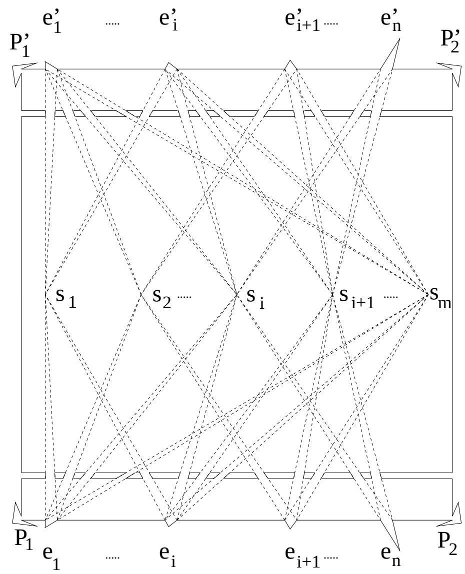

Fig. 3. Final construction

4 Properties of the Reduction

4.1 The Reduction is Feasible

In order to make the reduction work, we request that at no point a guard sees three or more spikes except if it is placed at some s_i. A guard that is placed at some point with y-value between 0 and b, i.e., between the barrier and the spikes, sees at most one spike, provided the barrier is placed such that no cones of two different elements intersect in the area below the barrier. A guard that is placed at some point with y-value between $b + b'$ and y_0, but not equal to y_0, sees at most two spikes, provided that the spikes are placed such that no three cones intersect in the area above the barrier, and provided that the view of the guard is blocked by the barrier as described. A guard with y-value greater than y_0 does not see any of the spikes at $y = 0$ since the view is blocked by the barrier. A guard that is placed at some point with y-value less than 0, covers at most one spike, if it is ensured that no two spikes intersect. Thus, we need to prove the following:

- No three cones from different elements intersect.
- The barrier is such that all intersections of cones from the same element e_i are below b and such that all intersections of cones from different elements are above $b + b'$ and such that all of the barrier except for the walls at $y = b + b'$ is visible from at least one of two guards at P_1 and P_2.
- No two spikes intersect.

No Three Cones from Different Elements Intersect

Lemma 1. *For* $e_l \in s_{l'}$, *let:*

$$D_l \geq \max\left(\frac{s_{i'} - s_{l'}}{s_{i'} - s_{j'}}(D_j + d - D_i) + D_i + d\right)$$

where the maximum is taken over all $e_i \in s_{i'}$ *and* $e_j \in s_{j'}$, *for which* $i < j < l$ *and* $l' < j' < i'$ *holds. Then the three cones from* e_l *to* $s_{l'}$, *from* e_i *to* $s_{i'}$ *and from* e_j *to* $s_{j'}$, *with* $i < j < l$ *do not have a common intersection point.*

Proof. Assume that the positions of the elements, i.e., the values D_v, have been set for all $v < l$ such that no three cones intersect. We show how to set D_l such that no three cones intersect; see Fig. 4. Let S be an intersection point with maximum y-value among any two cones of elements to the left of e_l. For each set $s_{l'}$ of which e_l is a member, draw a line through S, determine where it intersects the line $y = 0$, and let $D^S_{l,l'}$ be the x-value of this intersection point. Let $D^S_l = \max_{l'} D^S_{l,l'}$ be the maximum x-value of all intersection points defined this way. For any pair of cones in "inverse position" to the left of e_l, with which a cone at e_l forms a "triple inversion", compute the corresponding D^S_l and let $D^{\max}_l$ be the maximum D^S_l. Finally, we let $D_l = D^{\max}_l + d$ to ensure that no three cones have one common intersection point at some point S. Figure 4 shows the situation for an intersection and explains the notation. The

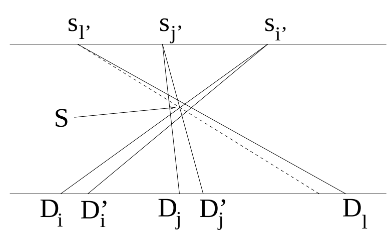

Fig. 4. Intersection of three cones

point S is the intersection point of the lines g_1 from $s_{i'}$ to D_i and g_2 from $s_{j'}$ to D'_j. Simple geometric calculations yield: $S = ((1 - t_1)s_{i'} + t_1 D_i, y_0(1 - t_1))$ with $t_1 = \frac{s_{i'} - s_{j'}}{D'_j - D_i + s_{i'} - s_{j'}}$. Let g_3 be the line from $s_{l'}$ to S, and simple geometric calculations show: $D^S_{l,l'} = \frac{s_{i'} - s_{l'}}{s_{i'} - s_{j'}}(D_j + d - D_i) + D_i$. The lemma follows. □

Lemma 1 implies:

$$\max\left(\frac{s_{i'} - s_{l'}}{s_{i'} - s_{j'}}(D_j + d - D_i) + D_i + d\right) \leq \max\left(\frac{md'}{d'}(D_j + d) + d\right) \;, \forall j < l$$
$$\leq m(D_{l-1} + d) + d$$

Now, let $D_l = m(D_{l-1} + d) + d$. It is easy to see that this is consistent with our definition of D_l, since: $d + 2d \sum_{j=1}^{l} m^j = m((d + 2d \sum_{j=1}^{l-1} m^j) + d) + d$

The Barrier is in Good Position

Lemma 2. *Any two cones that belong to the same element e_i intersect only at points with y-values at most $y_0 \frac{d}{d+d'}$.*

Proof. Let e_i be a member of s_j and s_l, and let $s_j < s_l$. The intersection point of the lines g_j from s_j to D'_i and g_l from s_l to D_i is the point in the intersection area of the two cones that has the largest y-value. The lemma follows by simple geometric calculations. □

Lemma 3. *Any two cones that belong to elements e_i, e_j, respectively, with $i < j$, intersect only at points with y-values at least $y_0 \frac{d_i}{d_i + md'}$.*

Proof. Let $e_i \in s_{i'}$ and let $e_j \in s_{j'}$. Furthermore, let $D_i < D_j$ and $s_{j'} < s_{i'}$. This is the exact condition for the corresponding two cones to intersect. The intersection point of the lines g_1 from $s_{j'}$ to D_j and g_2 from $s_{i'}$ to D'_i is the point in the intersection area of the two cones with minimum y-value. The lemma follows by simple geometric calculations. □

Lemma 4. *Let $b' = \frac{bd(y_0 - b)}{y_0(p_2 - p_1) - d(y_0 - b)}$, where p_1 and p_2 are the x-values of the points P_1 and P_2. Then all of the barrier including the segments of the cones except for the segments at $y = b + b'$ are visible from the two guards at P_1 and P_2.*

Proof. Let $e_i \in s_j$ and let G_1 and G_2 be the two points where the corresponding cone intersects with the barrier line $y = b$ (see Fig. 2). We find an expression for y_1, which is the y-value of the intersection point of the two lines from P_1 to G_1 and from P_2 to G_2, and the lemma follows by simple geometric calculations. □

If we substitute $b = \frac{5}{12} y_0$ and $p_2 - p_1 = d + 2d \sum_{i=1}^{n} m^i + d'' - (-d'') = d + 2d \sum_{i=1}^{n} m^i + 2d''$ in the equation for b', we obtain $b' = \frac{\frac{35}{12^2} y_0}{2 \sum_{i=1}^{n} m^i + 2\frac{d''}{d} - \frac{7}{12}}$. A simple calculation shows that $b' < \frac{y_0}{12}$, if $m \geq 2$ and $n \geq 2$, which must be the case since there were no intersections otherwise.

Because of $d = \frac{d'}{2}$ and because of Lemma 2, any two cones from the same element intersect only at points with y-value at most $\frac{1}{3} y_0$, which is less than b. Because of $d_i \geq md'$ for all d_i and because of Lemma 3, any two cones from different elements intersect only at points with y-value at least $\frac{1}{2} y_0$, which is at most $b + b'$.

Spikes of Two Elements Do Not Intersect

Lemma 5. *The spikes of any two elements do not intersect.*

Proof. We determine the x-value x_l of the point I_l in the spike of e_l. Note that $x_l > D_l$. Simple calculations show that $x_l \leq 2D_l$. Since $D_{l+1} = m(D_l + d) + d$ and since we can assume that $m \geq 2$, the lemma follows. □

4.2 The Reduction Preserves Optimality

In this section we will show how our reduction maps solutions of the TC-instance $f(I)$ to solutions of the SC-instance I. We prove that optimum solutions are mapped to optimum solutions by showing each of both directions in a lemma.

Lemma 6. *If there exists a feasible solution of the SC-instance I with k sets, then there exists a feasible solution of the TC-instance $f(I)$ with $k+4$ guards.*

Proof. For each set s_j in the solution of the SC-instance, place a guard at height h at point s_j, and place four additional guards at height h at the points P_1, P_2, P_1', P_2'. □

Lemma 7. *If there exists a feasible solution of the TC-instance $f(I)$ with $k+4$ guards, then there exists a feasible solution of the SC-instance I with k sets.*

Proof. We describe the function g that maps a solution for TC to a solution for SC. Given a solution of the TC-instance $f(I)$, proceed as follows: Move the guard that covers point A (at height 0) of the ear at P_1 (see Fig. 2) to P_1. For the remaining three ears, proceed accordingly.

Observe that a guard that covers the spike of some element e_i must lie in a cone that leads from this spike to some point s_j. For each spike, there must be at least one guard that completely covers the spike, since the spike is one triangle in the terrain. Move each guard that lies in only one cone (i.e. not in an intersection area of several cones) to the endpoint s_j of the cone. Move all guards that lie in an area where at least two cones intersect and that are below the barrier line $y = b$ (or above the barrier line $y = 2y_0 - b$) to the endpoint of s_j of any of the intersecting cones.

For guards that lie in an intersection of two cones from different elements e_q, e_r, proceed as follows: Note that in this case we have one guard available to cover two elements. Determine how the spikes e_q' and e_r' of the mirror image are covered. If they are covered by a guard that lies in an intersection of two cones from e_q' and e_r', we have two guards available to cover two elements and the problem is resolved by moving one of the two guards available to any s_j of which e_q is a member, and by moving the other guard to any s_i of which e_r is a member. If e_q' is covered by a guard that lies in an intersection of the two cones of e_q' and some $e_{q'}'$ and if $e_{q'}'$ is also covered by a guard at some s_i, then there are four guards available to cover four elements and the problem is resolved by moving the available guards to appropriate s_i's. If e_q' is only covered by a guard that lies in the intersection of two cones of e_q' and some $e_{q'}'$ that is not covered by a guard at any s_i and if e_r' is only covered by a guard that lies in the intersection of two cones of e_r' and some $e_{r'}'$ that is not covered by a guard at any s_i, then we let $M' = \{q, r\}$ and determine how the mirror images of $e_{q'}'$ and $e_{r'}'$, which are $e_{q'}$ and $e_{r'}$, are covered. If they are covered by a guard that lies in the intersection of two cones of $e_{q'}$ and $e_{r'}$, then we have four guards available to cover four elements and the problem is resolved by moving the available guards. If $e_{q'}$ is covered by a guard that lies in an intersection of two cones of $e_{q'}$ and some $e_{q''}$ and if $e_{q''}$ is also covered by a guard at some s_i or if $q'' \in M'$, then we have five guards available to cover five elements and the problem is resolved by moving the available guards. If e'_q is only covered by a guard that lies in the intersection of two cones of $e'_{q'}$ and some $e'_{q''}$ that is not covered by a guard at any s_i and if $e'_{r'}$ is only covered by a guard that lies in the intersection of two cones of $e'_{r'}$ and some $e'_{r''}$ that is not covered by a guard at any s_i, and if neither q'' nor r'' are in M', then we add q' and r' to M' and proceed accordingly for the mirror images of $e_{q''}$ and $e_{r''}$, which are $e'_{q''}$ and $e'_{r''}$. This procedure will stop after $n/2$ iterations at the latest, since two indices are added to M' in each step. After $n/2$ steps, the number of guards available will be greater or equal to the number of elements to be covered.

Guards that lie inside the polygon but outside the cones cannot cover any spikes completely and are therefore removed. Guards that lie outside the polygon are also removed.

This rearrangement of guards correctly guards the terrain without increasing the number of guards. A solution for the SC-instance can be determined by including set s_j in the SC solution if and only if there is a guard at point s_j. □

Lemmas 6 and 7 establish the following theorem:

Theorem 1. *An optimum solution of the SC-instance I contains k sets, if and only if an optimum solution of the TC-instance $f(I)$ contains $k+4$ guards.*

The description of the function g also shows that we are able to efficiently find an optimum solution of the SC-instance I, if we are given an optimum solution of the TC-instance $f(I)$.

4.3 The Reduction is Polynomial

Note that d, d', y_0, h, b are all constants in our reduction. The values for b' and for all D_i are computable in polynomial time and can be expressed with $O(n \log m)$ bits. Therefore, the function f runs in time polynomial in the size of the input SC-instance, since it only produces a polynomial number of triangles from which each corner can be computed in polynomial time and each corner takes at most $O(n \log m)$ bits to be expressed. It is obvious that the function g runs in polynomial time, since it only involves moving around a polynomial number of guards. If the number of guards is super-polynomial, we have an immediate transformation by selecting every set in the SC-instance. It takes polynomial time to move each guard, since it needs to be determined in which cone(s) a guard lies.

The polynomiality of the reduction and Theorem 1 establish the following corollary:

Corollary 1. TERRAIN COVER *is NP-hard.*

5 An Inapproximability Result

In order to get a strong inapproximability result, we take advantage of a property of the SC-instances produced in the reduction in [4], used to prove an optimum inapproximability result for SC.

Lemma 8. *Let N be the number of elements and let M be the number of sets in any SC-instance produced by the reduction in [4]. Then $M \leq N^5$ holds.*

Proof. $N = mR = m(5n)^l$ according to and adopting the notation of [4]. There are k provers. Each of them can be asked $Q = n^{\frac{l}{2}}(\frac{5n}{3})^{\frac{l}{2}}$ questions. An answer contains $\frac{l}{2} + \frac{3l}{2} = 2l$ bits, therefore there are 2^{2l} possible answers for each question. Since for each prover and each question/answer-pair a set is added, there are $M = k \cdot Q \cdot 2^{2l}$ sets. We prove that there is a constant t such that $N^t > M$, which is equivalent to $t > \frac{\log M}{\log N}$, where log denotes the base 2 logarithm. To do so, observe that $\frac{\log M}{\log N} = \frac{\log k + \log Q + 2l}{\log m + l \log 5 + l \log n}$. Since $Q = n^{\frac{l}{2}}(\frac{5n}{3})^{\frac{l}{2}} = (\frac{5n^2}{3})^{\frac{l}{2}}$, we get $\frac{\log M}{\log N} = \frac{\log k + \frac{l}{2} \log \frac{5}{3} + l \log n + 2l}{\log m + l \log 5 + l \log n}$. Since $m = n^{\Theta(l)}$, there must be a constant $c > 0$ with $m \leq n^{cl}$ for large enough values of l. Since $k < l$ we get $\frac{\log M}{\log N} = \frac{l(\frac{\log k}{l} + \frac{1}{2} \log \frac{5}{3} + \log n + 2)}{l(c \log n + \log 5 + \log n)} \leq \frac{1+1+2+\log n}{\log n}$. Since n is the number of variables in the input instance of 5-Occurrence-3-Sat, we can assume $n \geq 2$. Therefore, we get $\frac{\log M}{\log N} \leq 5$. □

Now consider only those SC-instances that are produced in the reduction in [4] and their corresponding TC-instances. Then, an approximation ratio of $(1 - \epsilon) \ln n$ for any $\epsilon > 0$ cannot be guaranteed by a polynomial algorithm for those SC-instances unless $NP \subseteq TIME(n^{O(\log \log n)})$, since this would imply that 5-Occurrence-3-Sat could be solved efficiently.

Theorem 2. *For all SC-instances I produced in the reduction in [4] and their corresponding TC-instances $f(I)$, there is a constant $c > 0$ such that, if TC for all considered instances can be approximated by a polynomial algorithm with an approximation ratio better than $c(1 - \epsilon) \ln |f(I)|$ for any $\epsilon > 0$, then SC for all considered instances can be approximated with an approximation ratio better than $(1 - \epsilon) \ln n$, where n is the number of elements in the SC-instance.*

Proof. If TC can be approximated with ratio better than $c(1 - \epsilon) \ln |f(I)|$, then we can find an approximate solution A' for each TC-instance that satisfies $\frac{|A'|}{|OPT'|} \leq c(1 - \epsilon) \ln |f(I)|$, where OPT' is an optimum solution of the TC-instance. Let $A = g(A')$ be the corresponding approximate solution for the SC-instance and let $OPT = g(OPT')$ be the optimum solution for the SC-instance. Because of Theorem 1 and the description of the function g, $|A'| = |A| + 4$ and $|OPT'| = |OPT| + 4$. We have $\frac{|A|+4}{|OPT|+4} \leq c(1 - \epsilon) \ln |f(I)|$ and therefore $\frac{|A|}{|OPT|} \leq c(1 - \epsilon) \ln |f(I)| + \frac{4}{|OPT|}(c(1 - \epsilon) \ln |f(I)|) - \frac{4}{|OPT|}$. With $|OPT| \geq 1$, we get $\frac{|A|}{|OPT|} \leq 5(c(1 - \epsilon) \ln |f(I)|)$ We need to express the number of triangles $|f(I)|$ of the TC-instance through the number of elements n in the SC-instance. Observe that the terrain of the TC-instance can always be triangulated such that the number of triangles is $O(nm)$. Therefore, $|f(I)| \leq nm\gamma$ for some constant γ. Because we can assume $\gamma < n$ and because of Lemma 8, we get $|f(I)| \leq nn^5 n = n^7$. (Note that if we had not restricted the set of possible SC-instances, then $m = 2^n$ would be possible and we would get a much weaker result.) Therefore, $\frac{|A|}{|OPT|} \leq 5 * 7c(1 - \epsilon) \ln n = 35c(1 - \epsilon) \ln n$. Thus, $c = \frac{1}{35}$. □

Thus, if TC could be approximated with a ratio $\frac{1-\epsilon}{35} \ln |f(I)|$, then SC could be approximated with a ratio $(1-\epsilon)\ln n$ for any $\epsilon > 0$. The contraposition of this sentence establishes our main result. Since SC cannot be approximated with a ratio $(1-\epsilon)\ln n$ according to [4], we get:

Theorem 3. *TC cannot be approximated by a polynomial time algorithm with an approximation ratio of* $\frac{1-\epsilon}{35} \ln n$ *for any* $\epsilon > 0$*, where* n *is the number of triangles, unless* $NP \subseteq TIME(n^{O(\log \log n)})$.

6 Conclusion

Theorem 3 together with our approximation algorithm with ratio $\ln n + 1$ settles the approximability of TERRAIN COVER up to a constant factor. It shows that TERRAIN COVER belongs to the relatively small family of NP-optimization problems with an approximation threshold of a non-trivial nature. Unfortunately, the approximation algorithm has an excessive running time and excessive space requirements, far too much for practical purposes if we take into account that the solution obtained might be far off the optimum. Therefore, it remains open how to solve the TERRAIN COVER problem in a practical situation. Our inapproximability result carries over to the problem of guarding a 2.5-dimensional terrain with guards on the terrain. As an aside, note that the restriction that each triangle must be covered completely by a single guard can be dropped without any consequences for the inapproximability result [3]. In that case, however, the proposed approximation algorithm cannot be applied.

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